

Time delays across saddles as a test of modified gravity

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Modified gravity theories can produce strong signals in the vicinity of the saddles of the total gravitational potential. In a sub-class of these models this translates into diverging time-delays for echoes crossing the saddles. Such models arise from the possibility that gravity might be infrared divergent or confined, and if suitably designed they are very difficult to rule out. We show that Lunar Laser Ranging during an eclipse could probe the time-delay effect within meters of the saddle, thereby proving or excluding these models. Very Large Baseline Interferometry, instead, could target delays across the Jupiter-Sun saddle. Such experiments would shed light on the infrared behaviour of gravity and examine the puzzling possibility that there might be well-hidden regions of strong gravity and even singularities inside the solar system.

Even though it is inevitable that Einstein’s theory of general relativity (GR) will not be the final word, it is telling that almost a century after its proposal all theories trying to supersede it have been ruled out or remain beyond detection [1–3]. Nonetheless, it is precisely the experimental misfortunes of “modified gravity” that prove the strength of GR, so it is important to keep pushing the boundaries, constructing and observationally disproving new possibilities. A further motivation derives from attempts to combine quantum theory and general relativity, a logical (if not an empirical) necessity. Such efforts invariably lead to corrections to GR, often at energy scales beyond the reach of current experiment, but not always. Finally, we should never forget that face value—taking into account only the matter sources that we do see—the observational status of GR in astrophysics and cosmology is calamitous. This is usually blamed on our imperfect knowledge of the matter content of the Universe, and dismissed by introducing new forms of invisible matter. But it could well be that the discrepancies signal a breakdown in our understanding of gravity.

Among the many theories attempting to extend GR some have tried to address the last issue, doing away with the need for non-visible or “dark” matter to explain anomalies at galactic, cluster and cosmological levels (see e.g. [4, 5]). Such theories have been labelled “MODified Newtonian Dynamics” (MOND), even though they have now been embedded into fully relativistic field theories (e.g. [6–12]). In all of them new effects are triggered below an acceleration scale, $a_0 \sim 10^{-10} \text{ ms}^{-2}$, a property suggested by the phenomenology. The fact that MONDian behaviour is physically triggered by an acceleration scale does not preclude writing a_0 in terms of length scale:

$$L_0 = \frac{c^2}{a_0} \quad (1)$$

for which a_0 could be a proxy. It is interesting that $L_0 \sim 30,000 \text{ Mpc}$ is of the order of the current horizon/Hubble radius. Nonetheless, it is the onset of low acceleration that triggers new effects. MONDian theories do not have preferred frames and do not break diffeomorphism invariance; yet new effects emerge in the

non-relativistic approximation when the *total* Newtonian force per unit mass falls below a_0 . For this reason one may expect the presence of “MONDian habitats” in the small regions encasing the saddle points of the gravitational potential in the Solar System, the points where the Newtonian force vanishes [13]. The prospect of a MONDian saddle test has motivated extensive work [14–19], with Lisa Pathfinder (LPF) in mind, but not only.

One of the most powerful tests of GR, by now elevated to the category of “classical” test, employs the echo time-delay effect. By flashing “light” (usually a radio wave) at a distant object, and catching its reflected “echo”, one measures a distinctive delay, if its path intersects a strong gravitational field. This so-called Shapiro effect was first detected with the radar echo off Venus in superior conjunction, leading to stringent constraints on the γ PPN parameter [1]. Since then the observational front has improved very fast. These tests use the fact that for a large class of theories (metric theories) the travel time is given by

$$t = \frac{1}{c} \int \left(1 - 2 \frac{\Phi}{c^2} \right) dz, \quad (2)$$

where Φ is the total gravitational potential, as obtained in the non-relativistic limit. Whilst the delay along a single path may be gauged away, the *variation* in delays along neighbouring paths is operationally meaningful, and constitutes a bona fide observational target.

Lunar Laser Ranging (LLR) is a major asset, among other fields, in gravitational physics (see for example [20]). Using lunar retroreflectors one may time very accurately echoes of sharp laser signals. It is immediately obvious that in principle LLR could probe the Moon saddle during a Lunar eclipse. Given that the Sun-Earth saddle is within the Lunar orbit, it turns out that it can also be targeted by LLR during a Solar eclipse. In practice a LLR saddle test requires the correct vantage point on Earth or even in its orbit. Very Large Baseline Interferometry (VLBI) is another asset in gravitational science (e.g. [21]), providing another strategy for probing delays across saddles. It correlates images of the same object (say, a quasar) as obtained in different continents.

A set up could be arranged in which one light ray goes through a saddle whilst the other (a few thousand kilometers away) does not. VLBI has the advantages that it could be used to probe other saddles, (e.g. the rather large Sun-Jupiter saddle), and that it relies only on the presence of a source behind the saddle, rather than on the vagaries of eclipses. In spite of potential difficulties, a saddle test with LLR or VLBI could be carried out without the logistic overheads associated with the LPF saddle extended mission.

It turns out that the time-delay effect is negligible for the MONDian models usually taken as targets for LPF. But there are also models well beyond the reach of LPF which predict a strong time-delay signal. Therefore the experimental test examined in this paper is complementary to a LPF test. This is hardly surprising. LPF accelerometers are sensitive to tidal stresses, i.e. the *second derivatives* of the gravitational potential (they feel their variation on a given frequency range). A time-delay test, instead, is sensitive to the *integral* of the potential along the line of sight (or rather, to its variation across neighbouring paths). It is therefore natural to find complementarity between the two measurements. An extension to LPF would constrain modified gravity theories which do away with the need for dark matter. A time-delay measurement would instead probe theories which encode the property that gravity is subject to confinement, as we shall now see.

It has long been speculated that gravity might resemble QCD in some aspects. Specifically, gravity could be asymptotically free (or “safe”) and reciprocally be subject to confinement, with a divergent strength at large distance, or low energy. The latter is actually what happens in the presence of a negative cosmological constant, inducing an attractive force satisfying Hook’s law (diverging like r). But more generally the issue has been raised in the context of the renormalization group flow of quantum general relativity (e.g. [22–24]), where it has been conjectured that the degenerate fixed point, when lifted, contains divergent IR behaviour. An equivalent implementation is possible in the context of MONDian theories, should these be released from their obligations as dark matter alternatives, but with a key property kept and exacerbated. Existing MONDian theories already have the peculiarity that they enhance the strength of gravity in situations where the standard Newtonian force would become weak. Specifically, for astrophysical applications, as $F_N \ll a_0$, we have that the MONDian force goes like $F_\phi \sim \sqrt{F_N}$, at least in spherically symmetric situations. Thus, the MONDian force still drops to zero with F_N , albeit slower. But what if it diverged instead? For example, we could imagine the “dual” behaviour $F_\phi \propto 1/F_N$, and once this possibility is considered we could consider sharper divergences, such as exponentials, or

$$F_\phi \sim \frac{1}{F_N^p} \quad (3)$$

with $p > 0$ very large. If one is to avoid appealing to dark matter clearly a_0 would have to be smaller than the usual one, but not otherwise. It turns out that it would be very difficult to rule out theories of this sort, except for their echo and VLBI saddle delays, as now show.

We first briefly lay down the formalism for defining MONDian theories, without wedding ourselves to a specific formulation. As explained in [14], in spite of the large number of MONDian theories, the expression for the non-relativistic potential invariably satisfies 3 types of equations only (which may be formally reduced to 2). The dynamics may be written as resulting from the usual Newtonian potential Φ_N and a “fifth force” field, ϕ , responsible for MONDian effects, with total potential $\Phi = \Phi_N + \phi$. For “Type I” theories ϕ is ruled by a non-linear Poisson equation:

$$\nabla \cdot (\mu(z) \nabla \phi) = \kappa G \rho, \quad (4)$$

where, for convenience, we pick the argument of the free function μ as:

$$z = \frac{\kappa}{4\pi} \frac{|\nabla \phi|}{a_0} \quad (5)$$

where κ is a dimensionless constant. For “Type II” theories we have instead:

$$\nabla^2 \phi = \frac{\kappa}{4\pi} \nabla \cdot (\nu(v) \nabla \Phi_N) \quad (6)$$

where the argument of free function ν is given by

$$v = \left(\frac{\kappa}{4\pi} \right)^2 \frac{|\nabla \Phi_N|}{a_0}. \quad (7)$$

Should these theories serve their duties as dark matter alternatives we should require $\mu \sim z$ for $z \ll 1$ and $\nu \sim 1/\sqrt{v}$ for $v \ll 1$, and a_0 should be the usual MONDian acceleration. However these theories have an interest in their own right: they may generally be regarded as theories with a preferred acceleration scale. More general functions μ or ν and values of a_0 should then be considered. If we want to keep the alternative to dark matter rationale, then the a_0 used here must be at least one order of magnitude smaller than that employed in traditional MONDian theories. However, if we detach these theories completely from that role, and if dark matter does exist and play a role in the dynamics, this is not true.

We illustrate our calculations using type II theories, because they’re simpler. For the purpose of investigating confined gravity we shall consider free functions which for $v \ll 1$ are power laws:

$$\nu \propto \frac{1}{v^n}. \quad (8)$$

From (6) and (3) we have $p = n - 1$ and so for $n > 1$ (to be contrasted with the MONDian $n = 1/2$) we obtain confinement behaviour. For $n = 3/2$ the theory mimics a negative cosmological constant in spherically symmetric

situations. The dual behaviour suggested in (3) with $p = 1$ follows from $n = 2$. Note that a negative Λ is the perfect dual to standard MOND behaviour. The potential ϕ diverges around a saddle if $n \geq 2$. However, this only translates into a divergent time-delay at the saddle if $n \geq 3$, with $n = 3$ representing a logarithmic divergence

Even though parameter κ will not appear in the final answer for the time-delay, it is important for justifying an approximation and setting a scale. The rationale for its appearance in (7) and proportionality constant in (6) is as follows [14]. If $\nu \rightarrow 1$ at large argument then Newton's constant G is renormalized by $\kappa/(4\pi)$, and this should be small. But so that $F_\phi \sim a_0$ when $F_N \sim a_0$, we should use (7) for argument of ν , if $\nu \sim 1/\sqrt{v}$ is to be triggered at $v \sim 1$ in the usual MONDian theory. With (8) the same requirement becomes:

$$v = \left(\frac{\kappa}{4\pi}\right)^{\frac{1}{n}} \frac{|\nabla\Phi_N|}{a_0}, \quad (9)$$

with $\nu \propto 1/v^n$ triggered at $v \sim 1$. The region where field ϕ goes strongly MONDian is now of size:

$$r_0 = \frac{a_0}{A} \left(\frac{\kappa}{4\pi}\right)^{\frac{1}{n}}. \quad (10)$$

However field ϕ is subdominant with respect to the Newtonian potential until we get to a distance:

$$\tilde{r} = \frac{a_0}{A} \quad (11)$$

from the saddle. It is only inside this inner bubble that ϕ is both MONDian and dominant.

For orientation purposes we first evaluate the delay effect in Newtonian theory. Introducing cylindrical coordinates, $\{z, \rho, \theta\}$, the delay is obtained by integrating Eq. (2) along paths of constant $\rho = b$. (This is valid for eclipse LLR only; the geometry is more complex for VLBI.) In the linear approximation the Newtonian potential is $\Phi^N = A\left(-\frac{z^2}{2} + \frac{\rho^2}{4}\right)$. For the Earth-Sun saddle the tidal stress is $A \approx 4.6 \times 10^{-11} \text{s}^{-2}$. Evaluating (2) over $z \in (-L/2, L/2)$, retaining only terms that vary with impact parameter b , leads to $\Delta t = -\frac{L}{c^3} \frac{Ab^2}{2}$. Even if the linear approximation were valid throughout the whole flight, we'd get variations of the order of 10^{-16}s for impact parameters $b \sim 1000 \text{km}$ (with a delay at the center with respect to outer trajectories). For the Jupiter saddle the effect is even smaller since $A \sim 1.8 \times 10^{-14} \text{s}^{-2}$. The Newtonian delay is therefore negligible.

We now repeat this calculation for type II theories, mimicking the calculation of ϕ in [17], for a free function of form (8). The potential satisfies ansatz:

$$\phi = -\frac{a_0^n}{A^{n-1}} \frac{1}{r^{n-2}} (f_0 + f_2 \cos(2\psi) + f_4 \cos(4\psi) + \dots) \quad (12)$$

where the parameters f have to be determined numerically. A particularly simple case follows from $n = 4$.

The integration should be performed along z within the region of strong MONDian behaviour, delimited by r_0 as defined above. For $n = 4$ the integration can be carried out explicitly, with leading order term in b/r_0 :

$$\Delta t \approx \frac{4a_0^4}{c^3 A^3} \frac{f_0}{b} \arctan \frac{r_0}{b}. \quad (13)$$

If we can assume that $r_0 \gg b$ (always true if $\kappa \ll 1$), this becomes asymptotically:

$$\Delta t \approx \frac{2\pi f_0}{b} \frac{a_0^4}{c^3 A^3}. \quad (14)$$

Parameter κ does not appear in the final answer, as long as it is small enough to justify taking $r_0/b \rightarrow \infty$. For more general n the calculation is more elaborate, leading to the asymptotic result ($b/r_0 \rightarrow 0$):

$$\Delta t \approx \frac{C}{b^{n-3}} \frac{a_0^n}{c^3 A^{n-1}}, \quad (15)$$

where C is given by:

$$C = \frac{2\sqrt{\pi}\Gamma\left(\frac{n-3}{2}\right)}{\Gamma\left(\frac{n-2}{2}\right)} \left(f_0 + \frac{n-4}{n-2}f_2 + \frac{(n-4)(n-6)}{n(n-2)}f_4\right).$$

As we see, $n = 4$ is a particularly simple limit of this expression. We note that for $n > 3$ the relative time variation at the saddle diverges, decreasing as a power-law in b as we move away from the “bull's eye”. In principle the constant C can be positive or negative, leading to a delay or an advance at the bull's eye, but we shall call it delay for definiteness.

In spite of this “divergence” the observational implications are less dramatic than might be expected. Using the cosmological length scale L_0 defined in Eqn.(1) and the strong MOND bubble scale \tilde{r} (in Eqn.(11)) we can rearrange Eq.(14) in the suggestive form:

$$\Delta t \approx C \frac{b}{c} \frac{\tilde{r}}{L_0} \left(\frac{\tilde{r}}{b}\right)^{n-2}. \quad (16)$$

The first factor (besides C) is just the time it takes to cross the region closest to the saddle. Unlike time-delays caused by the Sun this is small, because the distances involved are small: b should be smaller than \tilde{r} and $\tilde{r} \sim 2.2 \text{m}$ for the Earth-Sun saddle and $\tilde{r} \sim 5.5 \text{km}$ for the Jupiter saddle. In addition the second factor relates the MOND bubble size to the horizon scale, introducing a tiny factor. Therefore, even though the third factor predicts a divergence, this will happen very close to the saddle and be observable only for very steeply diverging functions.

We spell out this expectation in Figs. 1 and 2, which describes the situation for the Earth-Sun saddle (as a target for LLR during a Solar eclipse) and the Jupiter-Sun saddle (as a target for VLBI), respectively. In both of these figures we have plotted the (base 10) logarithm of the time-delay in picoseconds, as a function of the impact parameter b (in meters) and the exponent n used in

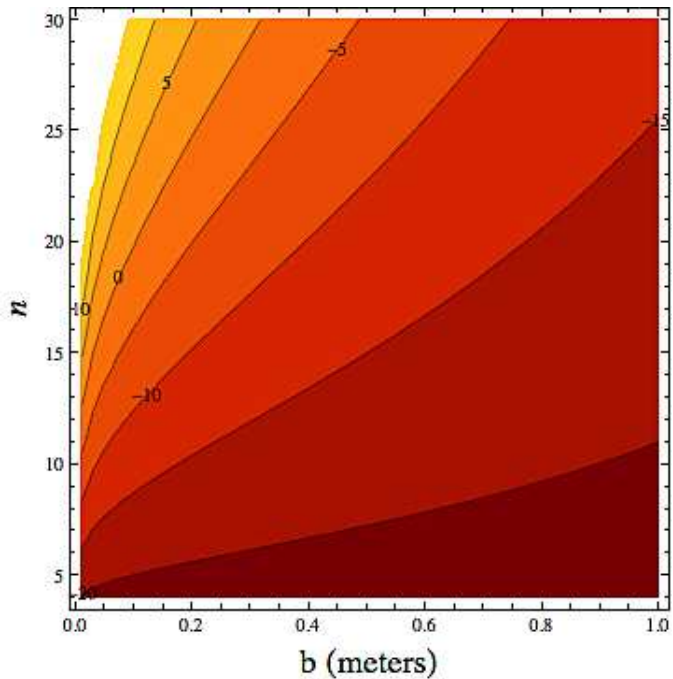


FIG. 1: The Log10 of the delay in picoseconds, as a function of impact parameter b and exponent n for the Earth-Sun saddle, as probed, say, by LLR during a Solar eclipse. As we can see the delay goes very quickly from very small to very large. Realistically, with current technology, only an integrated effect might be observable, and even then only for large n .

free-function ν . In both cases we observe a very abrupt transition from the very small to the very large, with the contour labeled zero denoting the rough borderline for observability with current technology. Typically the Earth saddle would have to be probed closer to a meter and even then assuming large values of n (in the range 20-30). The Jupiter saddle might be more forgiving, and small values of $n \sim 5$ could come within reach for b of the order of a meter, with $n \sim 20$ still constrained even for impacts of the order of a kilometer. In all fairness we cannot be overenthusiastic about the detectability of this effect in the first setting, where with current technology it would be seen at best as an integrated effect (the wavepackets often have a width of about 200 meters). The next generation of lunar retroreflectors could be necessary. The second situation might be more hopeful. We illustrated our conclusions with the Earth-Sun saddle during a Solar eclipse but similar results apply for the Moon saddle, as targeted by LLR during a Lunar eclipse. Likewise what we have shown for the Jupiter saddle has a closely related counterpart with the Saturn saddle. Incidentally, Eq. (16) can be used to prove that the effect for standard MONDian functions ($n = 1/2$) is negligible ($\Delta t \sim 10^{-34}$ s with b about a meter from the saddle). Likewise it can be shown that the functions considered here would have negligible effect for a LPF test.

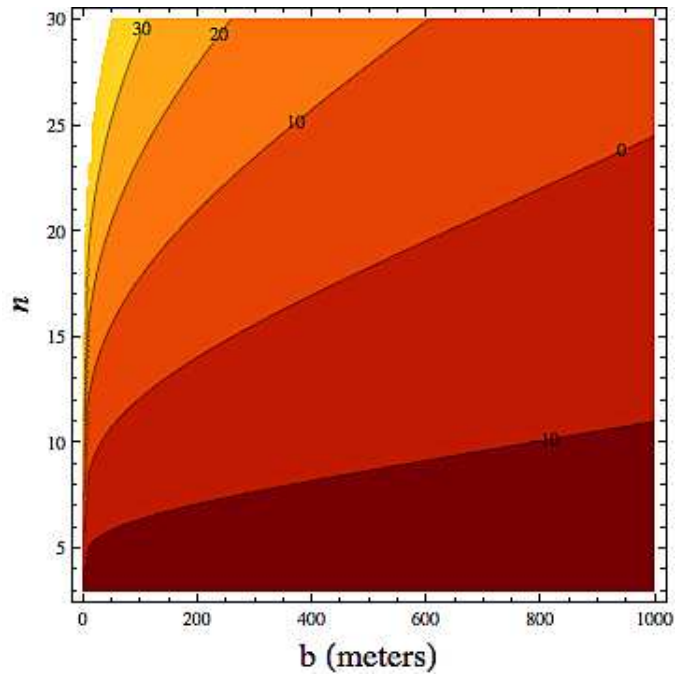


FIG. 2: The Log10 of the delay in picoseconds, as a function of impact parameter b and exponent n for the Jupiter-Sun saddle, as potentially probed by VLBI. Again the delay goes very quickly from very small to very large. Here the boundary is closer to realistic experimental parameters, and lower exponents n come within reach.

What about other MONDian theories? As an example, we briefly discuss what was labeled type I theories in [14]. For these the situation is more complex due to the well known presence of a curl field, softening the divergence [13, 14]. Under strict spherical symmetry this field vanishes, so although this is not applicable to a saddle, we can gain some intuition. Parametrizing the free-function in (4) as:

$$\mu \propto \frac{1}{z^m} \quad (17)$$

for $z \ll 1$, we see that *ignoring the curl field*, the exponent p in (3) is $p = 1/(m-1)$, so that $m > 1$ becomes the condition for confining behaviour. Now, $m = 3$ is equivalent to a negative cosmological constant Λ , and $m = 2$ leads to a perfect dual ($p = 1$). However, conclusions about the conditions for a divergence at the saddle are more subtle, because the magnetic field cannot be neglected. If this were the case, then $1 < m \leq 2$ would lead to a diverging ϕ , and $1 < m < 3/2$ to a diverging delay. However, now we can only try out the more flexible ansatz:

$$\phi = -C_1 \frac{1}{r^\alpha} (f_0 + f_2 \cos(2\psi) + f_4 \cos(4\psi) + \dots) \quad (18)$$

and search for a solution numerically (using techniques presented, in a different context, in [25]). We find for

$m = 1.1$, for example, that $\alpha = 1.25$ (instead of $\alpha = 9$, expected if the curl field could be neglected). Thus, these theories are even more difficult to constrain than type II. The situation is similar with type III (which also have a curl field).

It is interesting that something as dramatic as this divergence can be so elusive. Furthermore, we have only solved the problem to linear order, and many questions can be raised beyond the scope of the calculation presented in this paper. For example, if gravity is confined and infrared divergent, as envisaged here, could there be a singularity at the saddle? If so, would this singularity be naked, or rather, would there be a horizon? Whilst a positive answer to the first question is plausible, the answer to the second question is far from obvious. In both cases the detectability of the time-delay effect for the free functions used above, as calculated here, is unlikely to improve. The field is not attractive, so the usual arguments about accretion disks and X ray emission do not apply (even considering, e.g. the solar winds). One may think it odd that naked singularities or horizons could be floating around in the Solar system, but in practice the regions where such extreme behaviour is felt are very small, and they could pass unnoticed.

Of course one could fluff up the divergence region by introducing functions of the form:

$$\nu = \frac{1}{(v-1)^n} \quad (19)$$

in type II theories, for example. Then, the non-linear regime would be entered close to the ellipsoid $z^2 + \rho^2/2 = \tilde{r}^2$, and depending on the details of the full relativistic theory, this could signal the formation of a horizon or a naked singularity. Either way, assuming LLR geometry, any photons with $b < 2\tilde{r}$ would be lost, i.e. they would have an infinite time-delay. Close to the disk defined by $b = 2\tilde{r}$ the time-delay would diverge as:

$$\Delta t = C' \frac{\tilde{r}}{c L_0} \left(\frac{\tilde{r}}{b - 2\tilde{r}} \right)^{n-2} \quad (20)$$

(written in a format to allow easy comparison with (16)). We would now need to be glued to the “horizon” for the effect to be measurable, unless n is very large. However, we would also have a “black spot”, comprising the disk $b < 2\tilde{r}$. This rather extreme free-function is the only possibility we found for rendering these theories more tangible, and clearly a large a_0 is then promptly ruled out.

In summary, we hope that in this paper we have stressed the radical difference between the gravitational physics probed by LLR or VLBI on the one hand, and LPF on the other, regarding saddle points. With LPF one probes second derivatives of the potential, locally. With LLR and VLBI one probes the integral of the potential, at the end points, as a cumulative effect. Therefore with LPF, for standard MONDian functions, we find a distinctly changing tidal stress at the saddle (to be contrasted with an essentially DC Newtonian background). We cannot realistically get close to the saddle, but even far out we can expect signals with large SNRs. With a delay test we can potentially probe the region very close to the saddle; however, the predicted effects for standard MONDian functions are tiny. Nevertheless, we become sensitive to functions which diverge at low accelerations, associated with confinement and strong infrared behaviour for gravity. Such theories predict extreme behaviour very close to the saddle, raising the possibility of singularities. They are beyond the reach LPF and they do not purport to present an alternative to dark matter. However they have an interest in the own right, and are targets for a time-delay test as performed by the current or next generation of lunar retroreflectors, and by VLBI.

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